

Wave-wave interactions in stratified fluids: A comparison of approaches.

Yuri V Lvov¹, Kurt Polzin² and Naoto Yokoyama³

¹ Department of Mathematical Sciences, Rensselaer Polytechnic Institute, Troy NY 12180

² Woods Hole Oceanographic Institution, MS#21, Woods Hole, MA 02543

³ Department of Mechanical Engineering, Doshisha University, Kyotanabe, Kyoto 610-0394 JAPAN

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Abstract

Various approaches have been developed over the last four decades to characterize the magnitude of nonlinear interactions between triads of internal waves in stratified oceanic flows. The present manuscript compares some of these approaches and their predictions for internal wave nonlinearity parameter and Boltzman rate. We demonstrate that, for *resonant* triads in the limit of long internal waves in hydrostatic balance and in the absence of rotation, these various approaches predict *equivalent* rates of energy transfer between waves. However, with the inclusion of background rotation and off-resonant interactions, these approaches lead to qualitatively different predictions. In particular, a noncanonical approach in Lagrangian coordinates leads to higher levels of nonlinearity at high frequencies and large wavenumbers than a canonical approach in isopycnal coordinates.

1 Introduction

Wave-wave interactions in stratified oceanic flows have been a subject of intensive research in the last four decades. Of particular importance is the existence of a “universal” internal-wave spectrum, the Garrett and Munk spectrum. It is generally perceived that the existence of a universal spectrum is, at least in part and perhaps even primarily, the result of nonlinear interactions of waves with different wavenumbers. Due to the quadratic nonlinearity of the underlying primitive equations and the fact that the linear internal-wave dispersion relation can satisfy a three-wave resonance condition, waves interact in triads. Therefore the question arises: how strongly do waves with wavenumber \mathbf{p} interact with wavevectors \mathbf{p}_1 and \mathbf{p}_2 , where $\mathbf{p} = \mathbf{p}_1 + \mathbf{p}_2$? What are the oceanographic consequences of this interaction?

Various approaches have been developed to characterize the magnitude of such interactions (Hasselmann, 1966; Kenyon, 1966, 1968; McComas, 1975; Müller and Olbers, 1975; Olbers, 1974, 1976; Pelinovsky and Raevsky, 1977; McComas and Bretherton, 1977; Pomphrey *et al.*, 1980; Voronovich, 1979; Milder, 1982; Caillol and Zeitlin, 2000; Lvov and Tabak, 2001, 2004) (see Table 1 for a summary of the major distinctions.)

All these approaches represent various attempts to derive a closed equation representing the slow time evolution of the wave field’s wave action spectrum. Such an equation is called a *kinetic equation* (Zakharov *et al.*, 1992).

Table 1: A list of various kinetic equations. Results from Olbers (1976); McComas and Bretherton (1977); Pomphrey *et al.* (1980) are reviewed in Müller *et al.* (1986), who state that Olbers (1976), McComas and Bretherton (1977) and an unspecified Eulerian representation are consistent on the resonant manifold. Pomphrey *et al.* (1980) utilizes Langevin techniques to assess nonlinear transports. Müller *et al.* (1986) characterizes those Langevin results as being mutually consistent with the direct evaluations of kinetic equations presented in Olbers (1976); McComas and Bretherton (1977). Kenyon (1968) states (without detail) that Kenyon (1966) and Hasselmann (1966) give numerically similar results. A formulation in terms of discrete modes will typically permit an arbitrary buoyancy profile, but obtaining results requires specification of the profile. Of the discrete formulations, Pomphrey *et al.* (1980) use an exponential profile and the others assume a constant stratification rate. The kinetic equations marked by [†] are investigated in §4, while kinetic equations marked by [‡] are investigated further in §5.

source	coordinate system	vertical structure	rotation	hydro-static	special
Hasselmann (1966)	Lagrangian	discrete	no	no	
Kenyon (1966, 1968)	Eulerian	discrete	no	no	non-Hamiltonian
Müller and Olbers (1975) ^{†‡}	Lagrangian	cont.	yes	no	
McComas (1975, 1977)	Lagrangian	cont.	yes	yes	
Pelinovsky and Raevsky (1977)	Lagrangian	cont.	no	no	Clebsh
Voronovich (1979) [†]	Eulerian	cont.	no	no	Clebsh
Pomphrey <i>et al.</i> (1980)	Lagrangian	discrete	yes	no	Langevin
Milder (1982)	Isopycnal	n/a	no	no	
Caillol and Zeitlin (2000) [†]	Eulerian	cont.	no	no	non-Hamiltonian
Lvov and Tabak (2001) [†]	Isopycnal	cont.	no	yes	canonical
Lvov and Tabak (2004) [‡]	Isopycnal	cont.	yes	yes	canonical

In this manuscript we concentrate on four of these different versions of the internal-wave

kinetic equation:

- a noncanonical description using Lagrangian coordinates (Olbers, 1974, 1976; Müller and Olbers, 1975),
- a canonical Hamiltonian description using Clebsh variables in Eulerian coordinates (Voronovich, 1979),
- a dynamical derivation of a kinetic equation without use of Hamiltonian formalisms in Eulerian coordinates (Caillol and Zeitlin, 2000),
- a canonical Hamiltonian description in isopycnal coordinates (Lvov and Tabak, 2001, 2004).

Our intent is to compare these approaches, and in particular, compare the predictions for the wavenumber-dependent characteristic nonlinear time scale of the Garrett and Munk wave action spectrum. To achieve this goal, we give necessary background in Section 2, briefly review approaches of Table 1 in Section (3), and then we demonstrate in Section 4 that, under assumption of hydrostatic balance and under the assumption of *resonant* wave-wave interactions, the interaction matrices associated with the listed approaches are *equivalent*. While one, with sufficient experience, might regard this as an intuitive statement, it is far from trivial. We will then demonstrate in Section 5 that, if the assumption of resonant wave-wave interactions is relaxed, both quantitatively and qualitatively different transfer rates are predicted. In particular, we show that the Boltzman rate ϵ_p , defined below in (2) is, in fact, representation dependent for near-resonant interactions.

We have *not*, at this time, achieved a detailed mathematical understanding of how these differences arise and consequently do not digress into a detailed discussion of why, for example, the radius of convergence of two consecutive series expansions in one coordinate system differs so dramatically from a single series expansion in a different coordinate system. We conclude in Section 6.

2 Background

A kinetic equation is a closed equation for the time evolution of the wave action spectrum in a system of weakly interacting waves. It is usually derived as a central result of wave turbulence theory. The concepts of wave turbulence theory provide a fairly general framework for studying the statistical steady states in a large class of weakly interacting and weakly nonlinear many-body or many-wave systems. In its essence, classical weak turbulence theory (Zakharov *et al.*, 1992) is a perturbation expansion in the amplitude of the nonlinearity, yielding, at the leading order, linear waves, with amplitudes slowly modulated at higher orders by resonant nonlinear interactions. This modulation leads to a resonant redistribution

of the spectral energy density among space- and time-scales, and is described by a kinetic equation.

Typical assumptions needed for the derivation of kinetic equations are:

- Weak nonlinearity,
- Gaussian statistics of the interacting wave field in wavenumber space and
- Resonant wave-wave interactions.

The derivation of the kinetic equation for general nonlinear systems is well studied and understood, and thus will not be repeated here. Three wave kinetic equations take the form (Zakharov *et al.*, 1992; Lvov and Nazarenko, 2004; Lvov *et al.*, 1997):

$$\begin{aligned}
\frac{dn_{\mathbf{p}}}{dt} = & 4\pi \int |V_{\mathbf{p}_1, \mathbf{p}_2}^{\mathbf{p}}|^2 f_{p12} \delta_{\mathbf{p}-\mathbf{p}_1-\mathbf{p}_2} \delta(\omega_{\mathbf{p}} - \omega_{\mathbf{p}_1} - \omega_{\mathbf{p}_2}) d\mathbf{p}_{12} \\
& - 4\pi \int |V_{\mathbf{p}_2, \mathbf{p}}^{\mathbf{p}_1}|^2 f_{12p} \delta_{\mathbf{p}_1-\mathbf{p}_2-\mathbf{p}} \delta(\omega_{\mathbf{p}_1} - \omega_{\mathbf{p}_2} - \omega_{\mathbf{p}}) d\mathbf{p}_{12} \\
& - 4\pi \int |V_{\mathbf{p}, \mathbf{p}_1}^{\mathbf{p}_2}|^2 f_{2p1} \delta_{\mathbf{p}_2-\mathbf{p}-\mathbf{p}_1} \delta(\omega_{\mathbf{p}_2} - \omega_{\mathbf{p}} - \omega_{\mathbf{p}_1}) d\mathbf{p}_{12}, \\
& \text{with } f_{p12} = n_{\mathbf{p}_1} n_{\mathbf{p}_2} - n_{\mathbf{p}}(n_{\mathbf{p}_1} + n_{\mathbf{p}_2}).
\end{aligned} \tag{1}$$

Here $n_{\mathbf{p}} = n(\mathbf{p})$ is a three-dimensional wave action spectrum (spectral energy density divided by frequency) and the interacting wavevectors \mathbf{p} , \mathbf{p}_1 and \mathbf{p}_2 are given by

$$\mathbf{p} = (\mathbf{k}, m),$$

i.e. \mathbf{k} is the horizontal part of \mathbf{p} and m is its vertical component. We assume the wavevectors are signed variables and wave frequencies $\omega_{\mathbf{p}}$ are restricted to be positive. The magnitude of wave-wave interactions $V_{\mathbf{p}, \mathbf{p}_1}^{\mathbf{p}_2}$ is a matrix representation of the coupling between triad members. It serves as a multiplier in the nonlinear convolution term in what is now commonly called the Zakharov equation – equation in the Fourier space for the waves field variable. This is also an expression that multiplies the cubic convolution term in the three-wave Hamiltonian.

For internal waves in the ocean such kinetic equation was derived by the approaches in Table 1. The development of a kinetic equation is facilitated by transforming to canonical coordinates in a Hamiltonian framework, for which one can demonstrate that the symmetries and hence conservation principles of the original equation set in the spatial/temporal domains have been preserved in the spectral domain (e.g. Zakharov *et al.*, 1992). Finding canonical coordinates, however, can be highly nontrivial. Transformations to canonical coordinates have been found using Clebsch variables in Eulerian coordinates (Voronovich, 1979) and in isopycnal coordinates (Lvov and Tabak, 2001, 2004). Kinetic equations in Lagrangian

coordinates start by averaging the Lagrangian of the stratified fluid (averaging of the variational principle) and then transform the Lagrangian to the Hamiltonian. The Lagrangian coordinate kinetic equations considered here are noncanonical. We also note that it is possible to obtain a kinetic equation directly from the dynamical equations of motion, without the use of the Hamiltonian structure. Such an approach was executed by Caillol and Zeitlin (2000). The conservation properties of non-canonical and non-Hamiltonian representations are not guaranteed unless explicitly demonstrated. The issue of conservation properties is greatly compounded for non-canonical and non-Hamiltonian representations off the resonant manifold.

A typical restriction is to exclude interactions with potential vorticity carrying members of the fluid dynamical system, for which we refer the reader to Lelong and Riley (1991) and Caillol and Zeitlin (2000). Even in these extended analyses a plane wave formulation is assumed that eliminates the potential vorticity associated with a slowly varying wave-packet structure (Bühler and McIntyre, 2005; Polzin, 2008).

Note that the kinetic equation allows us to numerically estimate the life time of any given spectrum. In particular, we can define a wavenumber dependent nonlinear time scale proportional to the inverse Boltzman rate:

$$\tau_{\mathbf{p}}^{\text{NL}} = \frac{n_{\mathbf{p}}}{\dot{n}_{\mathbf{p}}} . \quad (2)$$

This time scale characterizes the net rate at which the spectrum changes and can be directly calculated from the kinetic equation.

One can also define the characteristic linear time scale,

$$\tau_{\mathbf{p}}^{\text{L}} = 2\pi/\omega_{\mathbf{p}}.$$

The non-dimensional ratio of these time scales can characterize the level of nonlinearity in the nonlinear system:

$$\epsilon_{\mathbf{p}} = \frac{\tau_{\mathbf{p}}^{\text{L}}}{\tau_{\mathbf{p}}^{\text{NL}}} = \frac{2\pi\dot{n}_{\mathbf{p}}}{n_{\mathbf{p}}\omega_{\mathbf{p}}} \quad (3)$$

We refer to (3) as the nonlinearity parameter.

The nonlinear parameter serves as a low order consistency check for the various kinetic equation derivations. An $O(1)$ value of $\epsilon_{\mathbf{p}}$ implies that the derivation of the kinetic equation is internally inconsistent. The Boltzman rate represents the net rate of transfer for wavenumber \mathbf{p} and is an appropriate measure of nonlinearity for smooth, isotropic and homogeneous spectra. The individual rates of transfer into and out of \mathbf{p} maybe significantly larger for spectral spikes (Müller *et al.*, 1986) and potentially for smooth, homogeneous but anisotropic spectra. Estimates of the Boltzman rate and $\epsilon_{\mathbf{p}}$ require integration of Eq. (1). In this manuscript such integration is performed numerically.

In this paper we concentrate on four approaches, namely Müller and Olbers (1975); Voronovich (1979); Caillol and Zeitlin (2000); Lvov and Tabak (2001, 2004). We show that on the resonant manifold they produce *equivalent* results.

Resonant interaction approximation is self-consistent for small level of nonlinearities. However, as the nonlinearity parameter increase, near-resonant interactions start to play a role.

For realistic estimates the effects of rotation must be included, and this restricts our investigations to two approaches that allow inclusion of background rotations. Therefore, we concentrate in more details on the Lvov and Tabak (2004) and Müller and Olbers (1975) representations.

We show that for the near-resonant interactions, these two approaches returns qualitatively different predictions for transfer rates. This is the main physical result of the present paper.

There is a multitude of reasons for possible differences. First and foremost, we view the distinction between Lagrangian, isopycnal and Eulerian coordinates as the most dynamically significant difference. The use of a Lagrangian coordinate system requires an expansion in powers of small fluid parcel displacements in addition to an assumption of weak nonlinearity, whereas formulations in isopycnal or Eulerian coordinates require only an assumption of weak nonlinearity. An issue with extant Lagrangian coordinate representations is that the small amplitude assumption represents an unconstrained approximation whose domain of validity *vis-a-vis* the weak interaction approximation is not well defined, (Müller *et al.*, 1986). A subsidiary issue is that the use of a Lagrangian coordinate system places the nonlinearity in the incompressibility constraint, and a single plane wave is not an exact solution of the equations of motion, (Sanderson, 1985). Similarly, a single plane wave also does not constitute a solution to the isopycnal equations of motion. In Eulerian coordinates the nonlinearity is advective and a single plane wave is an exact solution of the equations of motion. On the other hand, it is a robust observational fact that Eulerian frequency spectra at high vertical wavenumber are contaminated by vertical Doppler shifting: near-inertial frequency energy is Doppler shifted to higher frequency at approximately the same vertical wavelength. Use of an isopycnal coordinate system considerably reduces this artifact, (Sherman and Pinkel, 1991). Thus differences in the approaches may represent physical effects rather than technical issues such as the proper implementation of a potential vorticity conservation statement (Caillol and Zeitlin, 2000).

We emphasize that our intent is to estimate transport rates for various approaches within a common framework and to compare those results. Our goal is a qualitative physical explanation of the possible reasons for the similarities and differences rather than a quantitative analytical explanation of how those differences arise.

3 Various Approaches

In this section we list the approaches that we use. We do so for completion and to transfer everything to a uniform notation. Our attention is restricted to the hydrostatic balance case, for which

$$|\mathbf{k}| \ll |m|. \quad (4)$$

A minor detail is that the linear frequency has different algebraic representations in isopycnal and Cartesian coordinates. The Cartesian vertical wavenumber, k_z , and the density wavenumber, m , are related as $m = -g/(\rho_0 N^2)k_z$ where g is gravity, ρ is density with reference value ρ_0 , N is the buoyancy (Brunt–Väisälä) frequency and f is the Coriolis frequency. In isopycnal coordinates the dispersion relation is given by,

$$\omega(\mathbf{p}) = \sqrt{f^2 + \frac{g^2}{\rho_0^2 N^2} \frac{|\mathbf{k}|^2}{m^2}}. \quad (5)$$

In Cartesian coordinates,

$$\omega(\mathbf{p}) = \sqrt{f^2 + N^2 \frac{|\mathbf{k}|^2}{k_z^2}}. \quad (6)$$

In the limit of $f = 0$ these dispersion relations assume the form

$$\omega_{\mathbf{p}} \propto \frac{|\mathbf{k}|}{|m|} \propto \frac{|\mathbf{k}|}{|k_z|} \quad (7)$$

3.1 Kenyon and Hasselmann

The first kinetic equations for wave-wave interactions in a continuously stratified ocean appear in Kenyon (1966), Hasselmann (1966) and Kenyon (1968). Kenyon (1968) states (without detail) that Kenyon (1966) and Hasselmann (1966) give numerically similar results. We have found that Kenyon (1966) differs from the four approaches examined below on one of the resonant manifolds, but have not pursued the question further. It is possible this difference results from a typographical error in Kenyon (1966). We have not rederived this non-Hamiltonian representation and thus exclude it from this study.

3.2 Müller and Olbers

Matrix elements derived in Olbers (1974) are given by $|V_{\mathbf{p}_1, \mathbf{p}_2}^{\mathbf{p}}{}^{\text{MO}}|^2 = T^+/(4\pi)$ and $|V_{\mathbf{p}_2, \mathbf{p}}^{\mathbf{p}_1}{}^{\text{MO}}|^2 = T^-/(4\pi)$. We extracted T^\pm from the Appendix of Müller and Olbers (1975). In our notation, in the hydrostatic balance approximation, their matrix elements are given by

$$\begin{aligned}
|V_{\mathbf{p}_1, \mathbf{p}_2}^{\mathbf{p}}|^2 = & \frac{(N_0^2 - f^2)^2}{32\rho_0} \omega \omega_1 \omega_2 \left| \frac{|\mathbf{k}| |\mathbf{k}_1| |\mathbf{k}_2|}{\omega \omega_1 \omega_2 |\mathbf{p}| |\mathbf{p}_1| |\mathbf{p}_2|} \right. \\
& \left(- \frac{\left(-m_1 \frac{\mathbf{k}_1 \cdot \mathbf{k}_2 - i f \mathbf{k}_2 \cdot \mathbf{k}_1^\perp / \omega_1}{k_1^2} + m_2 \right) \left(-m_2 \frac{\mathbf{k}_1 \cdot \mathbf{k}_2 - i f \mathbf{k}_1 \cdot \mathbf{k}_2^\perp / \omega_2}{k_2^2} + m_1 \right)}{m} \right. \\
& \quad \left. - \frac{\left(-m_2 \frac{\mathbf{k}_2 \cdot \mathbf{k} + i f \mathbf{k}_2 \cdot \mathbf{k}^\perp / \omega_2}{k_2^2} + m \right) \left(-m \frac{\mathbf{k}_2 \cdot \mathbf{k} - i f \mathbf{k} \cdot \mathbf{k}_2^\perp / \omega}{k^2} + m_2 \right)}{m_1} \right. \\
& \quad \left. \left. - \frac{\left(-m \frac{\mathbf{k} \cdot \mathbf{k}_1 - i f \mathbf{k} \cdot \mathbf{k}_1^\perp / \omega}{k^2} + m_1 \right) \left(-m_1 \frac{\mathbf{k} \cdot \mathbf{k}_1 + i f \mathbf{k}_1 \cdot \mathbf{k}^\perp / \omega_1}{k_1^2} + m \right)}{m_2} \right) \right|^2. \tag{8}
\end{aligned}$$

Taking a $f = 0$ limit we get:

$$\begin{aligned}
|V_{\mathbf{p}_1, \mathbf{p}_2}^{\mathbf{p}}|^2 \propto & \frac{|\mathbf{k}| |\mathbf{k}_1| |\mathbf{k}_2|}{|m m_1 m_2|} \left(-\frac{1}{m} \left(-\frac{m_2 \mathbf{k}_1 \cdot \mathbf{k}_2}{|\mathbf{k}_2|^2} + m_1 \right) \left(-\frac{m_1 \mathbf{k}_2 \cdot \mathbf{k}_1}{|\mathbf{k}_1|^2} + m_2 \right) \right. \\
& \left. + \frac{1}{m_1} \left(\frac{m_2 \mathbf{k} \cdot \mathbf{k}_2}{|\mathbf{k}_2|^2} - m \right) \left(-\frac{m \mathbf{k}_2 \cdot \mathbf{k}}{|\mathbf{k}|^2} + m_2 \right) + \frac{1}{m_2} \left(-\frac{m \mathbf{k}_1 \cdot \mathbf{k}}{|\mathbf{k}|^2} + m_1 \right) \left(\frac{m_1 \mathbf{k} \cdot \mathbf{k}_1}{|\mathbf{k}_1|^2} - m \right) \right)^2 \tag{9}
\end{aligned}$$

3.3 Pelinovsky and Raevsky

An important paper on internal waves is Pelinovsky and Raevsky (1977). Clebsh variables are used to obtain the interaction matrix elements for both constant stratification rates, $N = \text{const.}$, and arbitrary buoyancy profiles, $N = N(z)$. Not much details are given, but there are some similarities in appearance with Voronovich (1979). The most significant result is the identification of a scale invariant (non-rotating, hydrostatic) stationary state. It is stated in the paper that their matrix elements are equivalent to those derived in their citation [11], which is Brehovsky (1975). Because Brehovsky (1975) and Pelinovsky and Raevsky (1977) are in Russian and not generally available, we refrain from including them in this comparison.

3.4 Voronovich

Voronovich used Clebsh-like variables to derive the Hamiltonian for incompressible stratified flows in the ocean. It is probably the first canonical Hamiltonian structure derived for such kind of flows. A detailed explanation of Voronovich's method appears in section 7.1 of the textbook Miropolsky (1981). It is a straightforward task to write down the kinetic equation associated with this Hamiltonian structure.

We formulate the matrix elements for Voronovich's Hamiltonian using his formula (A.1). This formula is derived for general boundary conditions. To compare with other matrix elements of this paper, we assume a constant stratification profile and Fourier basis as the vertical structure function $\phi(z)$. That allows us to solve for the matrix elements defined via Eq. (11) and above it in his paper. Then the convolutions of the basis functions give delta-functions in vertical wavenumbers. Voronovich's equation (A.1) transforms into:

$$|V_{\mathbf{p}_1, \mathbf{p}_2}^{\mathbf{p}}|^2 \propto \frac{|\mathbf{k}||\mathbf{k}_1||\mathbf{k}_2|}{|mm_1m_2|} \left(-m \left(\frac{1}{|\mathbf{k}||m|} \left(\frac{\mathbf{k} \cdot \mathbf{k}_1|m_1|}{|\mathbf{k}_1|} + \frac{\mathbf{k} \cdot \mathbf{k}_2|m_2|}{|\mathbf{k}_2|} \right) + \frac{\omega_1 + \omega_2 - \omega}{\omega} \right) \right. \\ \left. + m_1 \left(\frac{1}{|\mathbf{k}_1||m_1|} \left(\frac{\mathbf{k} \cdot \mathbf{k}_1|m|}{|\mathbf{k}|} + \frac{\mathbf{k}_1 \cdot \mathbf{k}_2|m_2|}{|\mathbf{k}_2|} \right) - \frac{\omega_1 + \omega_2 - \omega}{\omega_1} \right) \right. \\ \left. + m_2 \left(\frac{1}{|\mathbf{k}_2||m_2|} \left(\frac{\mathbf{k} \cdot \mathbf{k}_2|m|}{|\mathbf{k}|} + \frac{\mathbf{k}_2 \cdot \mathbf{k}_1|m_1|}{|\mathbf{k}_1|} \right) - \frac{\omega_1 + \omega_2 - \omega}{\omega_2} \right) \right)^2. \quad (10)$$

Note that Eq. (10) shares structural similarities with the interaction matrix elements in *isopycnal* coordinates, Eq. (13) below.

3.5 Milder

An alternative Hamiltonian description was developed in Milder (1982), in isopycnal coordinates without assuming a hydrostatic balance. The resulting Hamiltonian is an iterative expansion in powers of a small parameter, similar to the case of surface gravity waves. In principle, that approach may also be used to calculate wave-wave interaction amplitudes. Since those calculations were not done in Milder (1982), we do not pursue the comparison further.

3.6 Caillol and Zeitlin

A non-Hamiltonian kinetic equation for internal waves was derived in Caillol and Zeitlin (2000), Eq. (61). To make it appear equivalent to more traditional form of kinetic equation, as in Zakharov *et al.* (1992), we make a change of variables $\mathbf{l} \rightarrow -\mathbf{l}$ in the second line, and $\mathbf{k} \rightarrow -\mathbf{k}$ in the third line of (61) of Caillol and Zeitlin (2000). If we further assume that all spectra are symmetric, $n(-\mathbf{p}) = n(\mathbf{p})$, then the kinetic equation assumes traditional form, as in Eq. (1), see Müller and Olbers (1975); Zakharov *et al.* (1992); Lvov and Tabak (2001, 2004).

The matrix elements according to Caillol and Zeitlin (2000) are shown as $X_{k,l,p}$ and $Y_{k,l,p}^{\pm}$ in Eqs. (62) and (63), where $|V_{\mathbf{p}_1, \mathbf{p}_2}^{\mathbf{p}}|^2 = X_{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}}$ and $|V_{\mathbf{p}_2, \mathbf{p}}^{\mathbf{p}_1}|^2 = Y_{\mathbf{p}_1, -\mathbf{p}_2, \mathbf{p}}^+$. In our notation

it reads

$$|V_{\mathbf{p}_1, \mathbf{p}_2}^{\mathbf{p}}{}^{\text{CZ}}|^2 \propto (|\mathbf{k}| \text{sgn}(m) + |\mathbf{k}_1| \text{sgn}(m_1) + |\mathbf{k}_2| \text{sgn}(m_2))^2 \frac{(m^2 - m_1 m_2)^2}{|m| |m_1| |m_2| |\mathbf{k}| |\mathbf{k}_1| |\mathbf{k}_2|} \\ \times \left(\frac{|\mathbf{k}|^2 - |\mathbf{k}_1| \text{sgn}(m_1) |\mathbf{k}_2| \text{sgn}(m_2)}{m^2 - m_1 m_2} m - \frac{|\mathbf{k}_1|^2}{m_1} - \frac{|\mathbf{k}_2|^2}{m_2} \right)^2. \quad (11)$$

3.7 Isopycnal Hamiltonian

Finally, in Lvov and Tabak (2004) the following wave-wave interaction matrix element was derived based on a canonical Hamiltonian formulation in isopycnal coordinates:

$$|V_{1,2}^{0 \text{ H}}|^2 = \frac{N^2}{32g} \left(\left(\frac{k \mathbf{k}_1 \cdot \mathbf{k}_2}{k_1 k_2} \sqrt{\frac{\omega_1 \omega_2}{\omega}} + \frac{k_1 \mathbf{k}_2 \cdot \mathbf{k}}{k_2 k} \sqrt{\frac{\omega_2 \omega}{\omega_1}} + \frac{k_2 \mathbf{k} \cdot \mathbf{k}_1}{k k_1} \sqrt{\frac{\omega \omega_1}{\omega_2}} \right. \right. \\ \left. \left. + \frac{f^2}{\sqrt{\omega \omega_1 \omega_2}} \frac{k_1^2 \mathbf{k}_2 \cdot \mathbf{k} - k_2^2 \mathbf{k} \cdot \mathbf{k}_1 - k^2 \mathbf{k}_1 \cdot \mathbf{k}_2}{k k_1 k_2} \right)^2 \right. \\ \left. + \left(f \frac{\mathbf{k}_1 \cdot \mathbf{k}_2^\perp}{k k_1 k_2} \left(\sqrt{\frac{\omega}{\omega_1 \omega_2}} (k_1^2 - k_2^2) - \sqrt{\frac{\omega_1}{\omega_2 \omega}} (k_2^2 - k^2) - \sqrt{\frac{\omega_2}{\omega \omega_1}} (k^2 - k_1^2) \right) \right)^2 \right). \quad (12)$$

Lvov and Tabak (2001) is a rotationless limit of Lvov and Tabak (2004). Taking the $f \rightarrow 0$ limit, the Lvov and Tabak (2004) reduces to Lvov and Tabak (2001), and (12) reduces to

$$|V_{\mathbf{p}_1, \mathbf{p}_2}^{\mathbf{p}}{}^{\text{H}}|^2 \propto \frac{1}{|\mathbf{k}| |\mathbf{k}_1| |\mathbf{k}_2|} \left(|\mathbf{k}| |\mathbf{k}_1 \cdot \mathbf{k}_2| \sqrt{\left| \frac{m}{m_1 m_2} \right|} + |\mathbf{k}_1| |\mathbf{k}_2 \cdot \mathbf{k}| \sqrt{\left| \frac{m_1}{m_2 m} \right|} + |\mathbf{k}_2| |\mathbf{k} \cdot \mathbf{k}_1| \sqrt{\left| \frac{m_2}{m m_1} \right|} \right)^2. \quad (13)$$

Observe that in this form, these equations share structural similarities with Eq. (10).

In this section we gave brief review of the various approaches that were developed for describing wave-wave interactions of internal waves in the ocean. While this review is necessarily brief, this is the first time all these papers are cited together by a single manuscript.

4 Resonant wave-wave interactions

How one can compare the function of two vectors \mathbf{p}_1 and \mathbf{p}_2 , and their sum or difference? First one realizes that out of 6 components of \mathbf{p}_1 and \mathbf{p}_2 , only relative angles between wavevectors

enter into the equation for matrix elements. That is because the matrix elements depend on the inner products of wavevectors. The overall horizontal orientation of the wavevectors does not matter: relative angles can be determined from a triangle inequality and the magnitudes of the horizontal wavevectors \mathbf{k} , \mathbf{k}_1 and \mathbf{k}_2 . Thus the only needed components are $|\mathbf{k}|$, $|\mathbf{k}_1|$, $|\mathbf{k}_2|$, m and m_1 (m_2 is computed from m and m_1). Further note that in the $f = 0$ and hydrostatic limit, all matrix elements become scale invariant functions. That is to say that it is sufficient to choose an arbitrary scalar value for $|\mathbf{k}|$, and m , since only $|\mathbf{k}_1|/|\mathbf{k}|$, $|\mathbf{k}_2|/|\mathbf{k}|$ and m_1/m enter the expressions for matrix elements. We make the particular (arbitrary) choice that $|\mathbf{k}| = m = 1$ for the purpose of numerical evaluation, and thus the only independent variables to consider are $|\mathbf{k}_1|$, $|\mathbf{k}_2|$ and m_1 . Finally, m_1 is determined from the resonance conditions, as explained in the next subsection below. As a result, we are left with a matrix element as a function of only two parameters, k_1 and k_2 . This allows us to easily compare the values of matrix elements on the resonant manifold.

4.1 Reduction to the Resonant Manifold

When confined to the traditional form of the kinetic equation, wave-wave interactions (scattering) are constrained to the resonant manifolds defined by

$$a) \quad \begin{cases} \mathbf{p} = \mathbf{p}_1 + \mathbf{p}_2 \\ \omega = \omega_1 + \omega_2 \end{cases} \quad b) \quad \begin{cases} \mathbf{p}_1 = \mathbf{p}_2 + \mathbf{p} \\ \omega_1 = \omega_2 + \omega \end{cases} \quad c) \quad \begin{cases} \mathbf{p}_2 = \mathbf{p} + \mathbf{p}_1 \\ \omega_2 = \omega + \omega_1 \end{cases} . \quad (14)$$

To compare matrix elements on the resonant manifold we are going to use the above resonant conditions and the internal-wave dispersion relation (7). To determine vertical components m_1 and m_2 of the interacting wavevectors, one has to solve the resulting quadratic equations. Without restricting generality we choose $m > 0$. There are two solutions for m_1 and m_2 given below for each of the three resonance types described above.

Resonances of type (14a) give

$$\begin{cases} m_1 = \frac{m}{2|\mathbf{k}|} \left(|\mathbf{k}| + |\mathbf{k}_1| + |\mathbf{k}_2| + \sqrt{(|\mathbf{k}| + |\mathbf{k}_1| + |\mathbf{k}_2|)^2 - 4|\mathbf{k}||\mathbf{k}_1|} \right) \\ m_2 = m - m_1. \end{cases} , \quad (15a)$$

$$\begin{cases} m_1 = \frac{m}{2|\mathbf{k}|} \left(|\mathbf{k}| - |\mathbf{k}_1| - |\mathbf{k}_2| - \sqrt{(|\mathbf{k}| - |\mathbf{k}_1| - |\mathbf{k}_2|)^2 + 4|\mathbf{k}||\mathbf{k}_1|} \right) \\ m_2 = m - m_1. \end{cases} , \quad (15b)$$

Note that because of the symmetry, (15a) translates to (15b) if wavenumbers 1 and 2 are exchanged.

Resonances of type (14b) give

$$\begin{cases} m_2 = -\frac{m}{2|\mathbf{k}|} \left(|\mathbf{k}| - |\mathbf{k}_1| - |\mathbf{k}_2| + \sqrt{(|\mathbf{k}| - |\mathbf{k}_1| - |\mathbf{k}_2|)^2 + 4|\mathbf{k}||\mathbf{k}_2|} \right) \\ m_1 = m + m_2. \end{cases} , \quad (16a)$$

$$\begin{cases} m_2 = -\frac{m}{2|\mathbf{k}|} \left(|\mathbf{k}| + |\mathbf{k}_1| - |\mathbf{k}_2| + \sqrt{(|\mathbf{k}| + |\mathbf{k}_1| - |\mathbf{k}_2|)^2 + 4|\mathbf{k}||\mathbf{k}_2|} \right) \\ m_1 = m + m_2. \end{cases}, \quad (16b)$$

Resonances of type (14c) give

$$\begin{cases} m_1 = -\frac{m}{2|\mathbf{k}|} \left(|\mathbf{k}| - |\mathbf{k}_1| - |\mathbf{k}_2| + \sqrt{(|\mathbf{k}| - |\mathbf{k}_1| - |\mathbf{k}_2|)^2 + 4|\mathbf{k}||\mathbf{k}_1|} \right) \\ m_2 = m + m_1. \end{cases}, \quad (17a)$$

$$\begin{cases} m_1 = -\frac{m}{2|\mathbf{k}|} \left(|\mathbf{k}| - |\mathbf{k}_1| + |\mathbf{k}_2| + \sqrt{(|\mathbf{k}| - |\mathbf{k}_1| + |\mathbf{k}_2|)^2 + 4|\mathbf{k}||\mathbf{k}_1|} \right) \\ m_2 = m + m_1. \end{cases}. \quad (17b)$$

Because of the symmetries of the problem, (16a) is equivalent to (17a), and (16b) is equivalent to (17b) if wavenumbers 1 and 2 are exchanged.

4.2 Comparison of matrix elements

As explained above, we assume $f = 0$ and hydrostatic balance. Such a choice makes the matrix elements to be scale-invariant functions that depend only upon $|\mathbf{k}_1|$ and $|\mathbf{k}_2|$. As a consequence of the triangle inequality we need to consider matrix elements only within a “kinematic box” defined by

$$||\mathbf{k}_1| - |\mathbf{k}_2|| < |\mathbf{k}| < |\mathbf{k}_1| + |\mathbf{k}_2|.$$

The matrix elements will have different values depending on the dimensions so that isopycnal and Eulerian approaches will give different values (5)-(6). To address this issue in the simplest possible way, we multiply each matrix element by a dimensional number chosen so that all matrix elements are equivalent for some specific wavevector. In particular, we choose the scaling constant so that $|V(|\mathbf{k}_1| = 1, |\mathbf{k}_2| = 1)|^2 = 1$. This allows a transparent comparison without worrying about dimensional differences between various formulations.

4.2.1 Resonances of the “sum” type (14a)

Figure 1 presents the values of the matrix element $|V_{\mathbf{p}_1, \mathbf{p}_2}^{\mathbf{p}}(15b)|^2$ on the resonant sub-manifold given explicitly by (15b). All approaches give equivalent results. This is confirmed by plotting the relative ratio between these approaches, and it is given by numerical noise (not shown). The solution (15a) gives the same matrix elements but with $|\mathbf{k}_1|$ and $|\mathbf{k}_2|$ exchanged owing to their symmetries.

4.2.2 Resonances of the “difference” type (14b) and (14c)

We then turn our attention to resonances of “difference” type (14b) for which (14c) could be obtained by symmetrical exchange of the indices. All the matrix elements $|V_{\mathbf{p}_2, \mathbf{p}}^{\mathbf{p}_1}(16a)|^2$ on the

resonant sub-manifold (16a), are shown in Fig. 2. All the matrix elements are equivalent. The relative differences between different approaches are given by numerical noise (not shown). Finally, $|V_{\mathbf{p}_2, \mathbf{p}}^{\mathbf{p}_1}(16b)|^2$ on the resonant sub-manifold (16b) are shown in Fig. 3. Again, all the matrix elements are equivalent.

The solutions (17a) and (17b) give the same matrix elements but with $|\mathbf{k}_1|$ and $|\mathbf{k}_2|$ exchanged as the solutions (16a) and (16b) owing to their symmetries.

4.2.3 Special triads

Three simple interaction mechanisms are identified by McComas and Bretherton (1977) in the limit of an extreme scale separation. In this subsection we look in closer detail at these special limiting triads to confirm that all matrix elements are indeed asymptotically consistent. The limiting cases are:

- the vertical backscattering of a high-frequency wave by a low frequency wave of twice the vertical wavenumber into a second high-frequency wave of oppositely signed vertical wavenumber. This type of scattering is called elastic scattering (ES). The solution (15a) in the limit $|\mathbf{k}_1| \rightarrow 0$ corresponds to this type of special triad.
- The scattering of a high-frequency wave by a low-frequency, small-wavenumber wave into a second, nearly identical, high-frequency large-wavenumber wave. This type of scattering is called induced diffusion (ID). The solution (15b) in the limit that $|\mathbf{k}_1| \rightarrow 0$ corresponds to this type of special triad.
- The decay of a low wavenumber wave into two high vertical wavenumber waves of approximately one-half the frequency. This is called parametric subharmonic instability (PSI). The solution (16a) in the limit that $|\mathbf{k}_1| \rightarrow 0$ corresponds to this type of triad.

To study the detailed behavior of the matrix elements in the special triad cases, we choose to present the matrix elements along a straight line defined by

$$(|\mathbf{k}_1|, |\mathbf{k}_2|) = (\epsilon, \epsilon/3 + 1)|\mathbf{k}|.$$

This line originates from the corner of the kinematic box in Figs. 1–3 at $(|\mathbf{k}_1|, |\mathbf{k}_2|) = (0, |\mathbf{k}|)$ and has a slope of 1/3. The slope of this line is arbitrary. We could have taken $\epsilon/4$ or $\epsilon/2$. The matrix elements here are shown as functions of ϵ in Fig. 4. We see that all four approaches are again *equivalent* on the resonant manifold for the case of special triads.

In this section we demonstrated that all four approaches we considered produce *equivalent* results on the resonant manifold in the absence of background rotation. This statement is not trivial, given the different assumptions and coordinate systems that have been used for the various kinetic equation derivations.

5 Smearing of the resonance manifold and near-resonant Interactions

5.1 Nonlinear frequency renormalization as a result of nonlinear wave-wave interactions

Above we have compared the values of matrix element on the *resonant manifold*. The resonant interaction approximation is a mathematical simplification which reduces the complexity of the problem. In this subsection we examine transfers including near resonant interactions. Our interest in near-resonant interactions has significant physical motivations. For example a major unresolved issue is the importance of Doppler shifting (Polzin, 2004). Of particular interest here is the variable effects of Doppler shifting in different coordinate systems. The resonant interaction approximation assumes, perforce, an expansion in terms of a non-advected wavefield, with dispersion relation given by Eq. (5) or Eq. (6). In the limit of extreme time scale separation between high frequency waves and a low frequency background, one is tempted to replace the non-advected frequency by its Doppler shifted intrinsic frequency counterpart, $\omega \rightarrow \omega - \mathbf{k} \cdot \bar{\mathbf{u}}$, in which ω and \mathbf{k} are the frequency and wavevector of the high frequency wave and $\bar{\mathbf{u}}$ is the velocity field of the low frequency wavefield. This is the genesis of the eikonal approach (Müller *et al.*, 1986) to internal wave-wave interactions. Then the resonant approximation is self-consistent for small values of nonlinearities. Indeed, change in the wave amplitude will be small, and the Doppler shift cancels from the frequency delta function. Yet, as nonlinearity increases, the near-resonant interactions become more and more pronounced, consequently the issue of Doppler shifting more and more important. Furthermore, near-resonant interactions play a major role in numerical simulations on a discrete grids (Lvov *et al.*, 2006), for time evolution of discrete systems (Gershgorin *et al.*, 2007), in acoustic turbulence (Lvov *et al.*, 1997), surface gravity waves (Janssen, 2003; Yuen and Lake, 1982), and internal waves (Voronovich *et al.*, 2006; Annenkov and Shrira, V.I., 2006).

To take into account the effects of near-resonant interactions self-consistently, the energy conserving delta-functions in Eq. (1), $\delta(\omega_{\mathbf{p}} - \omega_{\mathbf{p}_1} - \omega_{\mathbf{p}_2})$, need to be “broadened”. The physical motivation for this broadening is the following: when the resonant kinetic equation is derived, it is assumed that the amplitude of each plane wave is constant in time, or, in other words, that the lifetime of single plane wave is infinite. Resulting kinetic equation, nevertheless, predicts that the amplitude of the wave do change. Consequently the wave lifetime is finite. For small level of nonlinearity this distinction is not significant, and resonant kinetic equation constitutes a self-consistent description. For larger values of nonliterary this is no longer the case, and the wave lifetime is finite and amplitude changes need to be taken into account. Consequently interactions may not be strictly resonant. This statement also follows from the Fourier uncertainty principle. In other words, the waves with varying

amplitude can not be represented by a single Fourier component. This effect is larger for stronger level of nonlinearity parameter.

Derivation of the kinetic equation with a broadened delta function is given in details in (Lvov *et al.*, 1997), and is not going to be repeated here. The result is that

$$\begin{aligned} \frac{dn_{\mathbf{p}}}{dt} = & 4 \int |V_{\mathbf{p}_1, \mathbf{p}_2}^{\mathbf{p}}|^2 f_{p12} \delta_{\mathbf{p}-\mathbf{p}_1-\mathbf{p}_2} \mathcal{L}(\omega_{\mathbf{p}} - \omega_{\mathbf{p}_1} - \omega_{\mathbf{p}_2}) d\mathbf{p}_{12} \\ & - 4 \int |V_{\mathbf{p}_2, \mathbf{p}}^{\mathbf{p}_1}|^2 f_{12p} \delta_{\mathbf{p}_1-\mathbf{p}_2-\mathbf{p}} \mathcal{L}(\omega_{\mathbf{p}_1} - \omega_{\mathbf{p}_2} - \omega_{\mathbf{p}}) d\mathbf{p}_{12} \\ & - 4 \int |V_{\mathbf{p}, \mathbf{p}_1}^{\mathbf{p}_2}|^2 f_{2p1} \delta_{\mathbf{p}_2-\mathbf{p}-\mathbf{p}_1} \mathcal{L}(\omega_{\mathbf{p}_2} - \omega_{\mathbf{p}} - \omega_{\mathbf{p}_1}) d\mathbf{p}_{12}, \end{aligned} \quad (18)$$

Here \mathcal{L} is defined as

$$\mathcal{L}(\Delta\omega) = \frac{\Gamma_{k12}}{(\Delta\omega)^2 + \Gamma_{k12}^2}, \quad (19)$$

where Γ_{k12} is the total broadening of each particular resonance, and is given below. If the nonlinear frequency renormalization tends to zero, i.e. $\Gamma_{k12} \rightarrow 0$, \mathcal{L} reduces to the delta function:

$$\lim_{\Gamma_{k12} \rightarrow 0} \mathcal{L}(\Delta\omega) = \pi \delta(\Delta\omega).$$

Consequently, in the limit resonant interactions (i.e. no broadening) (18) reduces to Eq. (1). We have shown in Lvov *et al.* (1997) that the broadening in Eq. (19) is given by

$$\Gamma_{k12} = \gamma_{\mathbf{p}} + \gamma_{\mathbf{p}_1} + \gamma_{\mathbf{p}_2}. \quad (20)$$

It means that the total resonance broadening is the sum of individual frequency broadening, and can be thus seen as the “triad interaction” frequency.

The single frequency renormalization can be calculated *self-consistently* from

$$\begin{aligned} \gamma_{\mathbf{p}} = & 4 \int |V_{\mathbf{p}_1, \mathbf{p}_2}^{\mathbf{p}}|^2 (n_{\mathbf{p}_1} + n_{\mathbf{p}_2}) \delta_{\mathbf{p}-\mathbf{p}_1-\mathbf{p}_2} \mathcal{L}(\omega_{\mathbf{p}} - \omega_{\mathbf{p}_1} - \omega_{\mathbf{p}_2}) d\mathbf{p}_{12} \\ & - 4 \int |V_{\mathbf{p}_2, \mathbf{p}}^{\mathbf{p}_1}|^2 (n_{\mathbf{p}_2} - n_{\mathbf{p}_1}) \delta_{\mathbf{p}_1-\mathbf{p}_2-\mathbf{p}} \mathcal{L}(\omega_{\mathbf{p}_1} - \omega_{\mathbf{p}_2} - \omega_{\mathbf{p}}) d\mathbf{p}_{12} \\ & - 4 \int |V_{\mathbf{p}, \mathbf{p}_1}^{\mathbf{p}_2}|^2 (n_{\mathbf{p}_1} - n_{\mathbf{p}_2}) \delta_{\mathbf{p}_2-\mathbf{p}-\mathbf{p}_1} \mathcal{L}(\omega_{\mathbf{p}_2} - \omega_{\mathbf{p}} - \omega_{\mathbf{p}_1}) d\mathbf{p}_{12}. \end{aligned} \quad (21)$$

The interpretation of this formula is the following: nonlinear wave-wave interactions lead to the change of wave amplitude, which in turn makes the lifetime of the waves to be finite. This, in turn, makes the interactions to be near-resonant.

A self-consistent estimate of $\gamma_{\mathbf{p}}$ requires the iterative solution of (18) and (21) over the entire field: the width of the resonance (21) depends on the lifetime of an individual wave [from (18)], which in turn depends on the width of the resonance (20). This numerically intensive computation is beyond the scope of this manuscript. Instead, we make the uncontrolled approximation that:

$$\gamma_{\mathbf{p}} = \delta\omega_{\mathbf{p}}. \quad (22)$$

We note that this choice is made for illustration purposes only, we certainly do not claim that it represents a self consistent choice. Below, we will take δ to be 10^{-2} and 10^{-3} . These values are rather small, therefore we remain in the closest proximity to the resonant interactions.

5.2 Characteristic nonlinear time scale of the Garrett and Munk Spectrum

Estimates of near-resonant transfers are obtained by assuming horizontal isotropy integrating (18) over horizontal azimuth:

$$\begin{aligned} \frac{\partial n_{\mathbf{p}}}{\partial t} = & 4\pi \int \frac{k_1 k_2}{S_{p12}} |V_{\mathbf{p}_1, \mathbf{p}_2}^{\mathbf{p}}|^2 f_{p12} \delta_{\mathbf{p}-\mathbf{p}_1-\mathbf{p}_2} \mathcal{L}(\omega_{\mathbf{p}} - \omega_{\mathbf{p}_1} - \omega_{\mathbf{p}_2}) dk_{12} dm_1 \\ & - 4\pi \int \frac{k_1 k_2}{S_{12p}} |V_{\mathbf{p}_2, \mathbf{p}}^{\mathbf{p}_1}|^2 f_{12p} \delta_{\mathbf{p}_1-\mathbf{p}_2-\mathbf{p}} \mathcal{L}(\omega_{\mathbf{p}_1} - \omega_{\mathbf{p}_2} - \omega_{\mathbf{p}}) dk_{12} dm_1 \\ & - 4\pi \int \frac{k_1 k_2}{S_{2p1}} |V_{\mathbf{p}, \mathbf{p}_1}^{\mathbf{p}_2}|^2 f_{2p1} \delta_{\mathbf{p}_2-\mathbf{p}-\mathbf{p}_1} \mathcal{L}(\omega_{\mathbf{p}_2} - \omega_{\mathbf{p}} - \omega_{\mathbf{p}_1}) dk_{12} dm_1, \end{aligned} \quad (23)$$

where S_{p12} is the area of the triangle $\mathbf{k} = \mathbf{k}_1 + \mathbf{k}_2$. We numerically integrated (23) for \mathbf{p} 's which have frequencies from f to N [specifically $(33/32, 17/16, 9/8, 5/4, 3/2, 2, 4, \dots)f$] and vertical wavenumbers from $4\pi/(2b)$ to $200\pi/(2b)$ ($[2, 4, 6, \dots, 98] \pi/b$). The limits of integration are restricted by horizontal wavenumbers from $2\pi/10^5$ to $2\pi/10$ meters $^{-1}$, vertical wavenumbers from $2\pi/(2b)$ to $2\pi/10$ meters $^{-1}$, and frequencies from f to N . The integrals over k_1 and k_2 are obtained in the kinematic box in $k_1 - k_2$ space. The grids in the $k_1 - k_2$ domain have 2^{17} points that are distributed heavily around the corner of the kinematic box. The integral over m_1 is obtained with 2^{13} grid points, which are also distributed heavily for the small vertical wavenumbers whose absolute values are less than $5m$, where m is the vertical wavenumber.

Below we calculate the nonlinear time scale (2) and nonlinearity parameter (3). To calculate this parameter, we need to choose a form of spectral energy density of internal waves. We utilize the Garrett and Munk spectrum as an agreed-upon representation of the internal waves:

$$E(\omega, m) = \frac{4f}{\pi^2 m_*} E_0 \frac{1}{1 + (\frac{m}{m_*})^2} \frac{1}{\omega \sqrt{\omega^2 - f^2}}. \quad (24)$$

Here the reference wavenumber is given by

$$m_* = \pi j_*/b, \quad (25)$$

in which the variable j represents the mode number of an ocean with an exponential buoyancy frequency profile having a scale height of b .

We choose the following set of parameters:

- $b = 1300$ m in the GM model
- The total energy is set as:

$$E_0 = 30 \times 10^{-4} \text{ m}^2 \text{ s}^{-2}.$$

- Inertial frequency is given by $f = 10^{-4}$ rad/sec, and buoyancy frequency is given by $N_0 = 5 \times 10^{-3}$ rad/sec.
- The reference density is taken to be $\rho_0 = 10^3$ kg/m³.

We then calculate the nonlinearity parameter (3) and the nonlinear time scale (2). To do so we substitute the Garrett and Munk spectrum (24) into the kinetic equation with broadening (18). For matrix elements we use Müller and Olbers (1975), Eq. (8), and Lvov and Tabak (2004), Eq. (12). We also use the dispersion relation of internal waves, (5) for the isopycnal Hamiltonian, and (6) for Lagrangian coordinates. We use two values of δ in (22): $\delta = 10^{-2}$ and $\delta = 10^{-3}$. We therefore make four calculations:

- Run I Lvov and Tabak (2004) with $\delta = 10^{-3}$
- Run II Müller and Olbers (1975) with $\delta = 10^{-3}$
- Run III Lvov and Tabak (2004) with $\delta = 10^{-2}$
- Run IV Müller and Olbers (1975) with $\delta = 10^{-2}$

Results appear in Figs. 5 and 6.

For Run 1 the nonlinearity parameter is uniformly small, smaller than 10^{-1} . Such value of the nonlinearity parameter indicates that the kinetic equation is a self-consistent approach for the Garrett and Munk Spectrum. Increasing values of the nonlinearity parameter are noted with increasing vertical wavenumbers. This is consistent with intuition that we have about such systems. The nonlinear time scale is of the order of one hundred wave periods at low vertical wavenumber and of order ten wave periods at high vertical wavenumber. We also define a “zero curve” - It is the locus of wavenumber-frequency where the nonlinearity parameter and time-derivative of waveaction is exactly zero. The zero curve clearly delineates a pattern of energy gain for frequencies $f < \omega < 2f$, energy loss for frequencies $2f < \omega < 5f$

and energy gain for frequencies $5f < \omega < N$. This seems to be a characteristic pattern that appears in our calculations. Note that the zero curves are nearly independent of vertical wavenumber.

The Müller and Olbers (1975), matrix element (8), Run II ($\delta = 10^{-3}$) results are qualitatively similar to Run I. Factor of 2-3 faster decorrelation times and levels of nonlinearity are noted in the high-frequency and high-wavenumber part of the spectrum.

Therefore we conclude that when near-resonant interactions are included, the transfer rates are representation dependent. Furthermore, Lagrangian approaches predict higher level of nonlinearity.

To investigate in more details results of near-resonant interactions, we perform numerical calculations for $\delta = 10^{-2}$. Results for the canonical Hamiltonian formulation in isopycnal coordinates Run III are nearly identical to those with $\delta = 10^{-3}$. Results for the Lagrangian coordinate representation are both *quantitatively* and *qualitatively* different. The Lagrangian coordinate formulation (8) now predicts $O(1)$ nonlinearity for high frequencies, while the isopycnal coordinate formulation still returns nonlinearity parameter and much slower decorrelation time estimates. The zero curves for the Lagrangian coordinate representation are no longer simple functions of frequency at this higher level of nonlinearity. The zero curves in the isopycnal coordinate system are relatively independent of δ .

To investigate the differences between approaches and the sensitivity of our results to the value of δ , in more details, we plot in Fig. 7 the differences of the nonlinearity parameter for these runs. In particular, we calculate the differences between Run I and Run II, Run I and Run III, and finally between Run II and Run IV. Differences associated with increased resonance broadening are minimal, 10^{-3} or smaller, for the isopycnal Hamiltonian. As the nonlinearity parameter estimates are representation dependent, differences between isopycnal coordinate and Lagrangian coordinate representations are much larger and increase with increasing δ .

We have found that transports for the canonical Hamiltonian representation are not too sensitive to near-resonant interactions. We have also found in Section 4 that all approaches are equivalent on the resonant manifold. We therefore conclude that all approaches will converge to Hamiltonian one as δ decreases. We have not undertaken such calculations as such small values of δ would require significant modifications to our numerical algorithm.

Note that the Fig. 6, especially Runs I and II, bear a strong resemblance to Fig. 4 of Olbers (1976). These figures contain two positive and one negative lobe with similar boundaries separating these regions, consistent with the characteristic pattern mentioned above. Olbers (1976) does not make the hydrostatic approximation, used the GM75 model as the basis of his evaluations and is constrained to the resonance manifold. We have made the hydrostatic approximation, base our evaluations on the GM76 model and have included resonance broadening. Similarities are also apparent with Fig. 12 of (McComas and Bretherton, 1977) and Figs 10 and 11 of McComas and Müller (1981). In those resonant evaluations

using the GM76 model, the hydrostatic approximation was invoked and interactions with frequencies greater than $N/3$ were excluded. The major difference is that the zero line separating the positive and negative lobe at high frequencies has moved to $10f$.

6 Discussion

In this paper we have reviewed different approaches for wave-wave interactions that have been presented in the literature in the last three decades. Namely, we have concentrated on the approaches of Müller and Olbers (1975); Voronovich (1979); Caillol and Zeitlin (2000); Lvov and Tabak (2001, 2004). In the absence of background rotation, we demonstrate that these four approaches produce *equivalent* results on the resonant manifold.

This statement is not trivial given the different assumptions and coordinate systems that have been used for the derivation of the various kinetic equations. It points to an internal consistency on the resonant manifold that we still do not completely understand and appreciate.

This result is less surprising for the canonical Hamiltonian approaches (Voronovich, 1979; Lvov and Tabak, 2001). A canonical Hamiltonian representation is the gold standard of wave turbulence. It guarantees that the symmetries and hence conservation principles of the original equation set in the spatial/temporal domains have been preserved in the spectral domain, (e.g. Zakharov *et al.*, 1992). Thus, if Voronovich's Clebsch variable representation in Eulerian coordinates and the Lvov and Tabak isopycnal Hamiltonian describe the same physical system, then there is a canonical transformation that connects these two Hamiltonians. It is well known that such a canonical transformation reduces to the identity transformation on the resonant manifold. To prove this statement one constructs a near-identical canonical transformation, which is applicable for weakly nonlinear systems (See Appendix A3 in Zakharov *et al.* (1992)). The Hamiltonian on the resonant manifold is invariant under a canonical near-identity transformation.

That is why Voronovich's matrix elements (10) look identical to the interaction matrix element in *isopycnal* coordinates (13) *on the resonant manifold*.

We can argue that, while the other two matrix elements (Caillol and Zeitlin (2000) and Müller and Olbers (1975)) are not in a canonical Hamiltonian formulation, they nevertheless do describe the same physical system. Consequently, they also can be approximated by a certain Hamiltonian structure, at least for small nonlinearities. This is explicitly the case for the noncanonical Hamiltonian of (Müller and Olbers, 1975). It appears to be implicitly true of the Caillol and Zeitlin (2000) non-Hamiltonian kinetic equation. Therefore equivalence of the scattering matrix element on the resonant manifold is an intuitive, yet not trivial statement.

On the other hand, it is also intuitive that there will be coordinate representation dependent differences. It is a robust observational fact that Eulerian frequency spectra at

high vertical wavenumber are contaminated by vertical Doppler shifting: near-inertial frequency energy is Doppler shifted to higher frequency at approximately the same vertical wavelength. Use of an isopycnal coordinate system considerably reduces this artifact (Sherman and Pinkel, 1991). Further differences are anticipated in a fully Lagrangian coordinate system (Pinkel, 2008). Thus differences in the approaches may represent physical effects and what is a stationary state in one coordinate system may not be a stationary state in another. In particular, differences may represent the effects of resonance broadening associated with Doppler shifting.

We also demonstrate that the isopycnal and Lagrangian coordinate system approaches predict qualitatively different results with the inclusion of the near-resonant interactions and background rotation. The canonical Hamiltonian isopycnal formalism is insensitive to off-resonant interactions: Broadening the resonance width by an order of magnitude does not create significant differences in the nonlinearity parameter. The noncanonical Lagrangian coordinate representation is, in contrast, quite sensitive to these changes.

As explained above, the Hamiltonian on the resonant manifold is invariant under near-identity canonical transformations. The kinetic equation describes the spectral transfers associated with the cubic terms of the Hamiltonian and conserves the energy associated with quadratic terms of the Hamiltonian. The kinetic equation should therefore return representation independent results on the resonant manifold. This statement is no longer true for near-resonant interactions.

Indeed, since the structure of the Hamiltonian may be altered off the resonant manifold by a near-identity canonical transformations, one *should* anticipate representation dependent differences in spectral energy transfer when near-resonant interactions are included. Such differences become more and more significant as nonlinearity increases and cubic parts of the Hamiltonian become increasingly large.

We would like to suggest that the differences between Müller and Olbers (1975) and Lvov and Tabak (2004) off the resonant manifold represent physical effects. However, an issue with extant Lagrangian coordinate representations is that they require a small amplitude assumption that represents an unconstrained approximation whose domain of validity *vis-a-vis* the weak interaction approximation is not well defined, (Müller *et al.*, 1986). On the basis of estimates of how horizontal Doppler shifting contributes to isopycnal spectra, we would anticipate that the Lagrangian coordinate stationary state would have typically steeper spectral slopes in the frequency domain than frequency spectra in isopycnal coordinates. The results presented here indicate that resonance broadening will quickly whiten the high frequency Lagrangian coordinate spectrum, in direct contradiction to our intuition regarding physical effects.

In this paper we have shown that while on the resonant manifold (i.e. for weakly nonlinear interactions) all approaches we considered do agree, inclusion of the near-resonant interactions (for stronger nonlinearities) should be done with care. Results with near resonant

interactions are representation dependent. This observations warrants further study.

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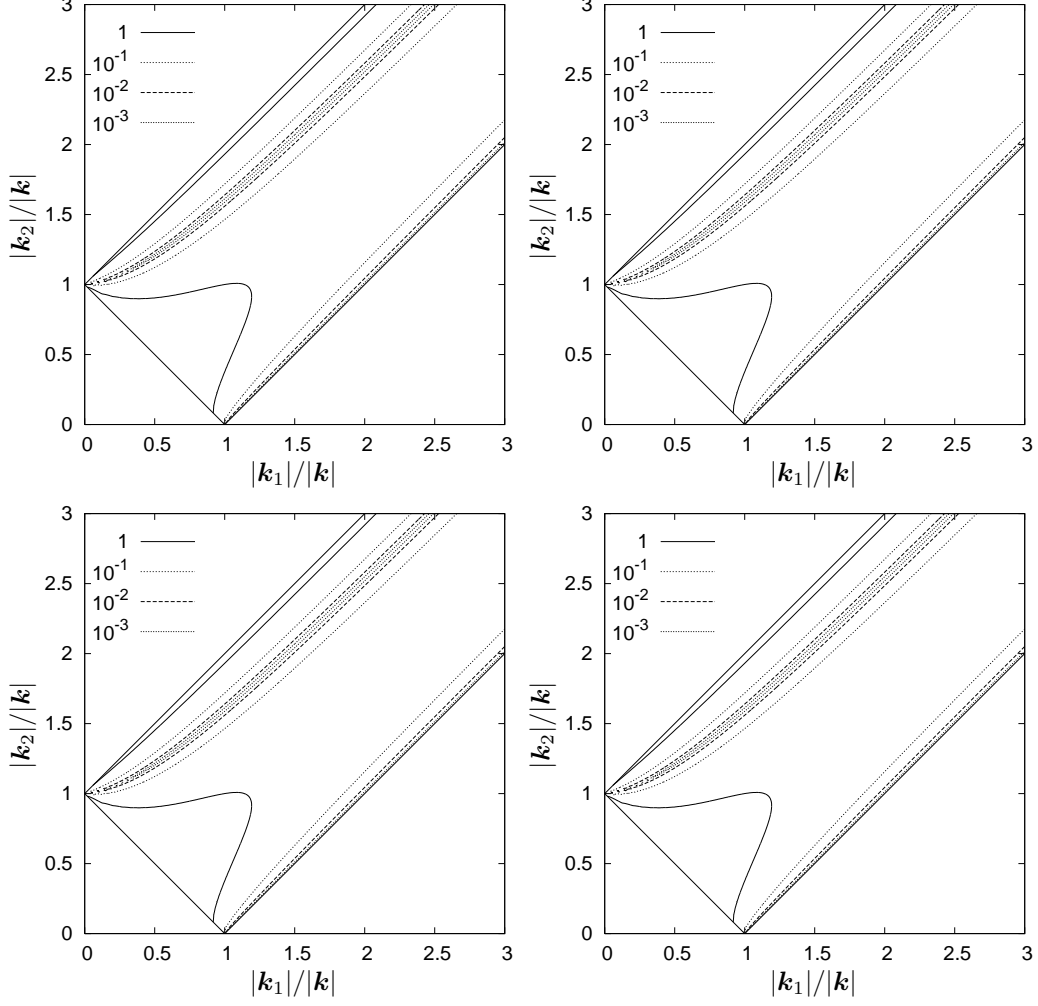


Figure 1: Matrix elements $|V_{\mathbf{p}_1, \mathbf{p}_2}^{\mathbf{p}}(15b)|^2$ given by the solution (15b). upper left: $|V_{\mathbf{p}_1, \mathbf{p}_2}^{\mathbf{p}, \text{MO}}(15b)|^2$ according to Müller and Olbers (1975), upper right: $|V_{\mathbf{p}_1, \mathbf{p}_2}^{\mathbf{p}, \text{V}}(15b)|^2$ according to Voronovich (1979), bottom left: $|V_{\mathbf{p}_1, \mathbf{p}_2}^{\mathbf{p}, \text{CZ}}(15b)|^2$ according to Caillol and Zeitlin (2000), bottom right: $|V_{\mathbf{p}_1, \mathbf{p}_2}^{\mathbf{p}, \text{H}}(15b)|^2$ according to Lvov and Tabak (2001).

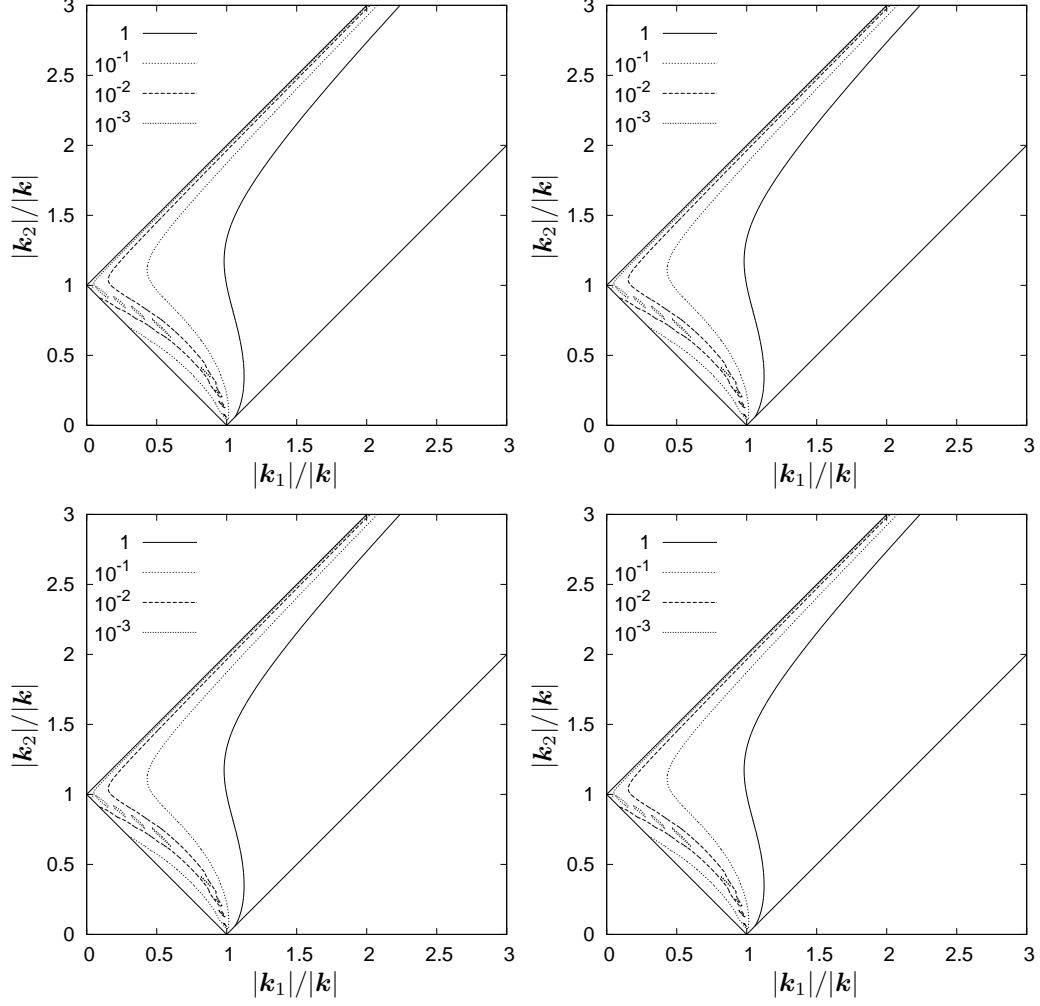


Figure 2: Matrix elements $|V_{\mathbf{p}_2, \mathbf{p}(16a)}^{\mathbf{p}_1}|^2$ given by the solution (16a). upper left: $|V_{\mathbf{p}_2, \mathbf{p}(16a)}^{\mathbf{p}_1 \text{ MO}}|^2$ according to Müller and Olbers (1975), upper right: $|V_{\mathbf{p}_2, \mathbf{p}(16a)}^{\mathbf{p}_1 \text{ V}}|^2$ according to Voronovich (1979), bottom left: $|V_{\mathbf{p}_2, \mathbf{p}(16a)}^{\mathbf{p}_1 \text{ CZ}}|^2$ according to Caillol and Zeitlin (2000), bottom right: $|V_{\mathbf{p}_2, \mathbf{p}(16a)}^{\mathbf{p}_1 \text{ H}}|^2$ according to Lvov and Tabak (2001).

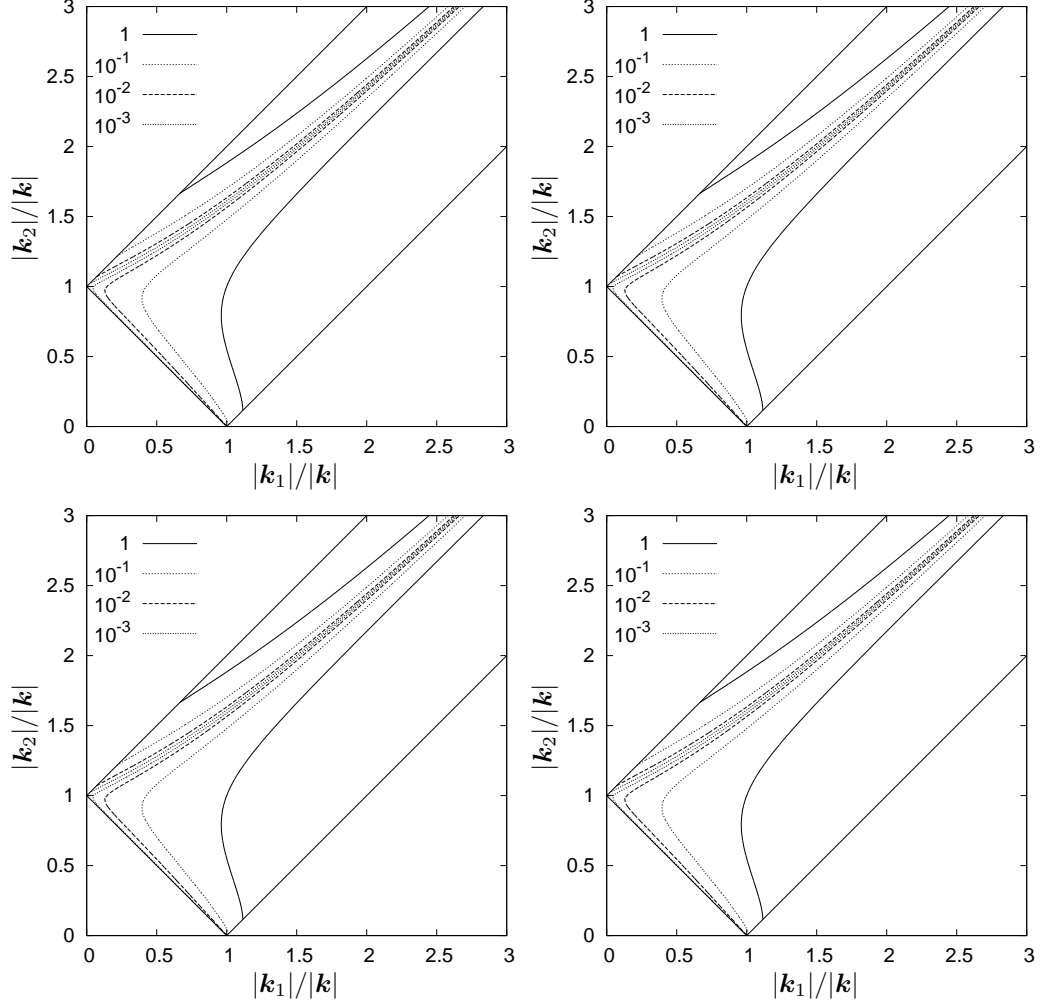


Figure 3: Matrix elements $|V_{\mathbf{p}_2, \mathbf{p}(16b)}^{\mathbf{p}_1}|^2$ given by the solution (16b). upper left: $|V_{\mathbf{p}_2, \mathbf{p}(16b)}^{\mathbf{p}_1 \text{ MO}}|^2$ according to Müller and Olbers (1975), upper right: $|V_{\mathbf{p}_2, \mathbf{p}(16b)}^{\mathbf{p}_1 \text{ V}}|^2$ according to Voronovich (1979), bottom left: $|V_{\mathbf{p}_2, \mathbf{p}(16b)}^{\mathbf{p}_1 \text{ CZ}}|^2$ according to Caillol and Zeitlin (2000), bottom right: $|V_{\mathbf{p}_2, \mathbf{p}(16b)}^{\mathbf{p}_1 \text{ H}}|^2$ according to Lvov and Tabak (2001).

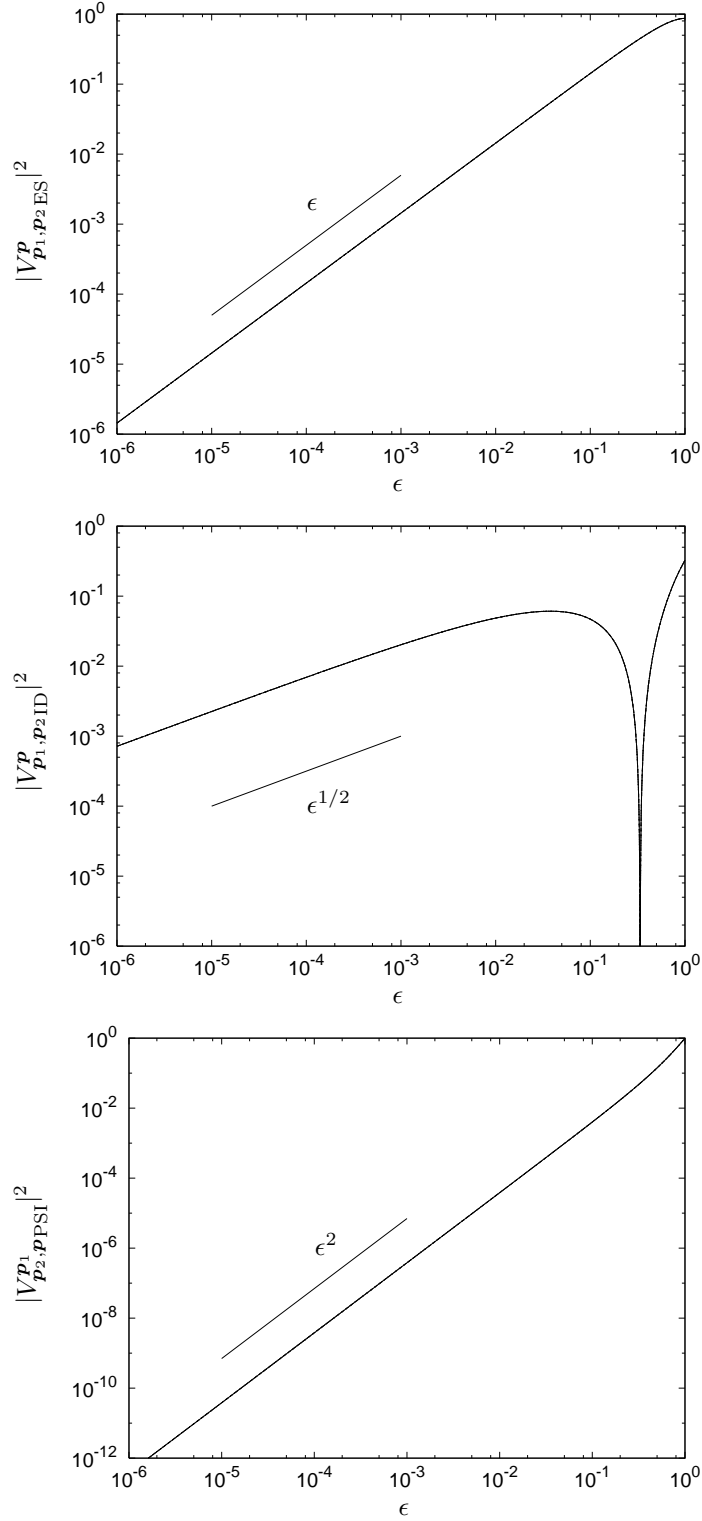


Figure 4: upper: Matrix elements $|V_{p_1, p_{2ES}}^p|^2$ given by the solution (15a). middle: Matrix elements $|V_{p_1, p_{2ID}}^p|^2$ given by the solution (15b). bottom: Matrix elements $|V_{p_2, p_{PSI}}^{p_1}|^2$ given by the solution (16a), which gives PSI as $|\mathbf{k}_1| \rightarrow 0$ ($\epsilon \rightarrow 0$). The matrix elements here are shown as functions of ϵ such that $(|\mathbf{k}_1|, |\mathbf{k}_2|) = \frac{1}{25}(\epsilon, \epsilon/3 + 1)|\mathbf{k}|$. All four versions of the Matrix elements are plotted here: the appearance of a single line in each figure panel testifies to the similarity of the elements on the resonant manifold.

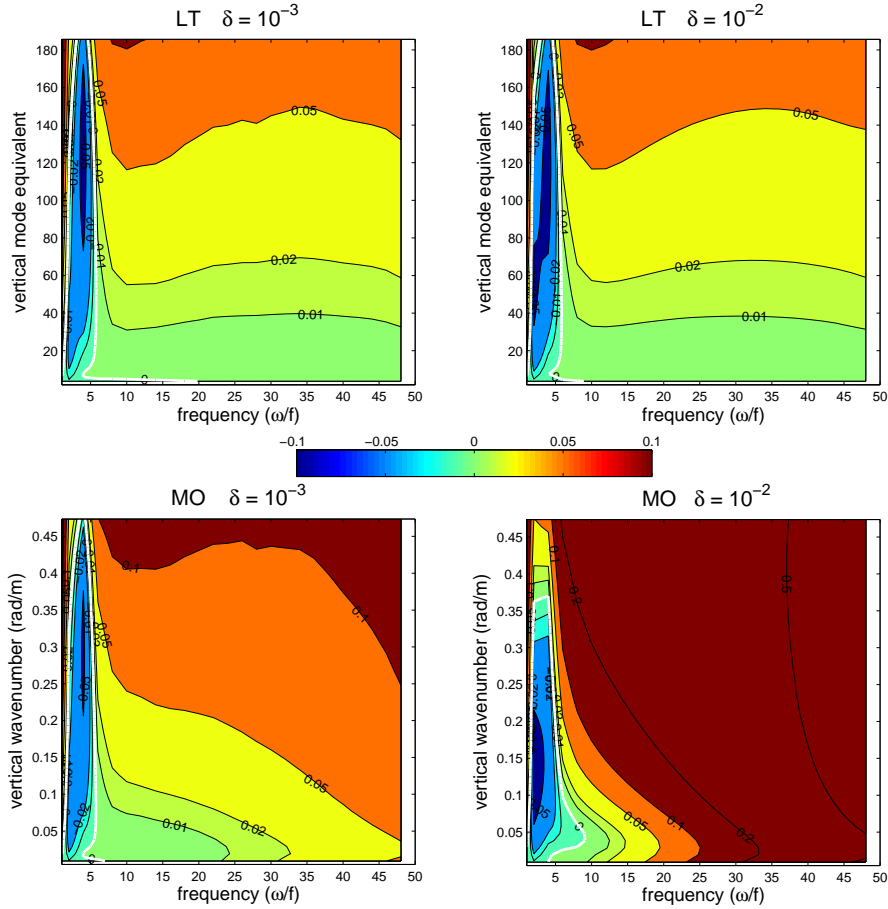


Figure 5: Nonlinearity parameter (3) for the Garrett and Munk spectrum (24) calculated via (18). The upper figures represent the value of nonlinearity parameter calculated using Lvov and Tabak (2004), equation (12) with $\delta = 10^{-3}$, Run I (upper left) and $\delta = 10^{-2}$ Run III (upper right). The bottom two pictures represent the value of nonlinearity parameter calculated via Müller and Olbers (1975), (8) with $\delta = 10^{-3}$ Run II (bottom left) and $\delta = 10^{-2}$ Run IV (bottom right).

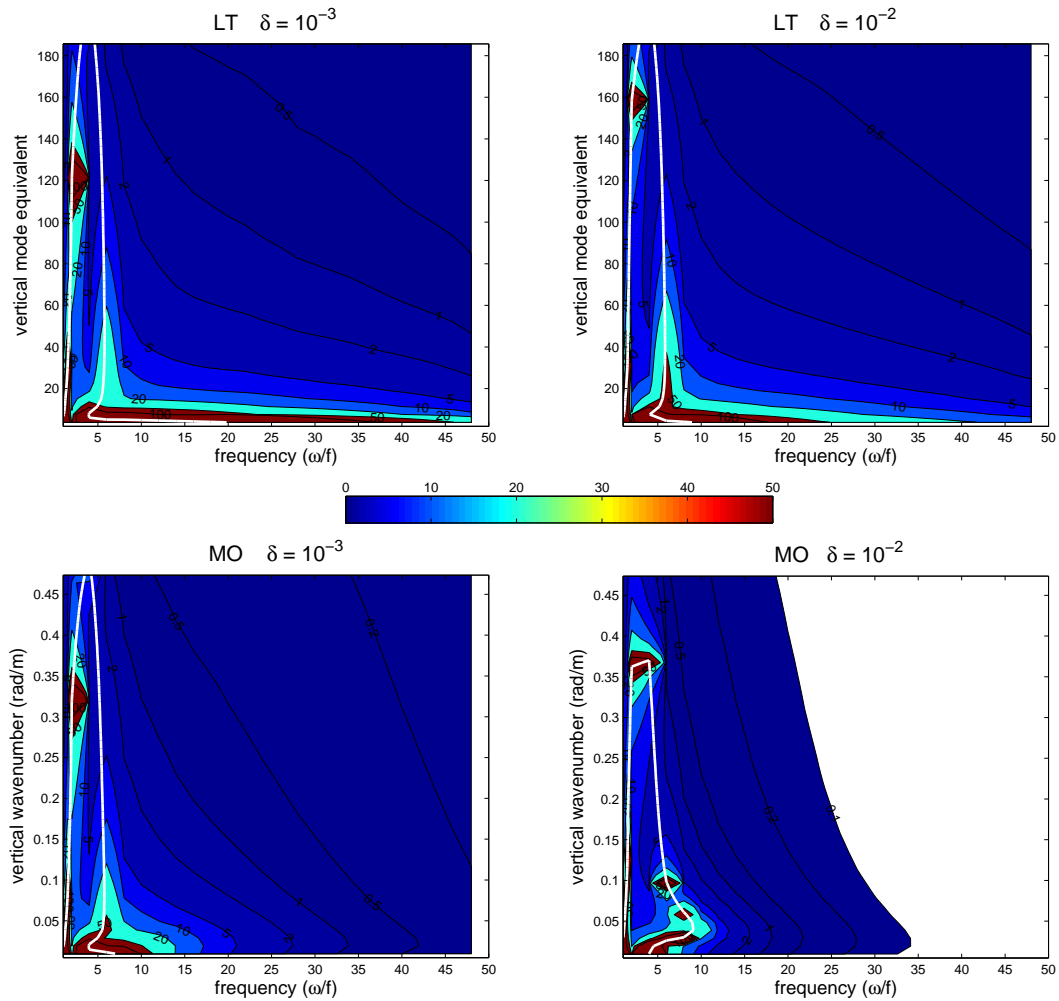


Figure 6: Nonlinear time (2) for the Garrett and Munk spectrum (24) calculated via (18). The upper figures represent the value of nonlinearity parameter calculated using Lvov and Tabak (2004), equation (12) with $\delta = 10^{-3}$, Run I (upper left) and $\delta = 10^{-2}$ Run III (upper right). The bottom two pictures represent the value of nonlinearity parameter calculated via Müller and Olbers (1975), (8) with $\delta = 10^{-3}$ Run II (bottom left) and $\delta = 10^{-2}$ Run IV (bottom right). On this bottom right figure white region to the left of the 0.1 contour corresponds to extremely fast time scales, faster than 0.1 of a day. On these figures, ω in cpd, m in cycle/m, and nonlinear time τ^{NL} is measured in days.

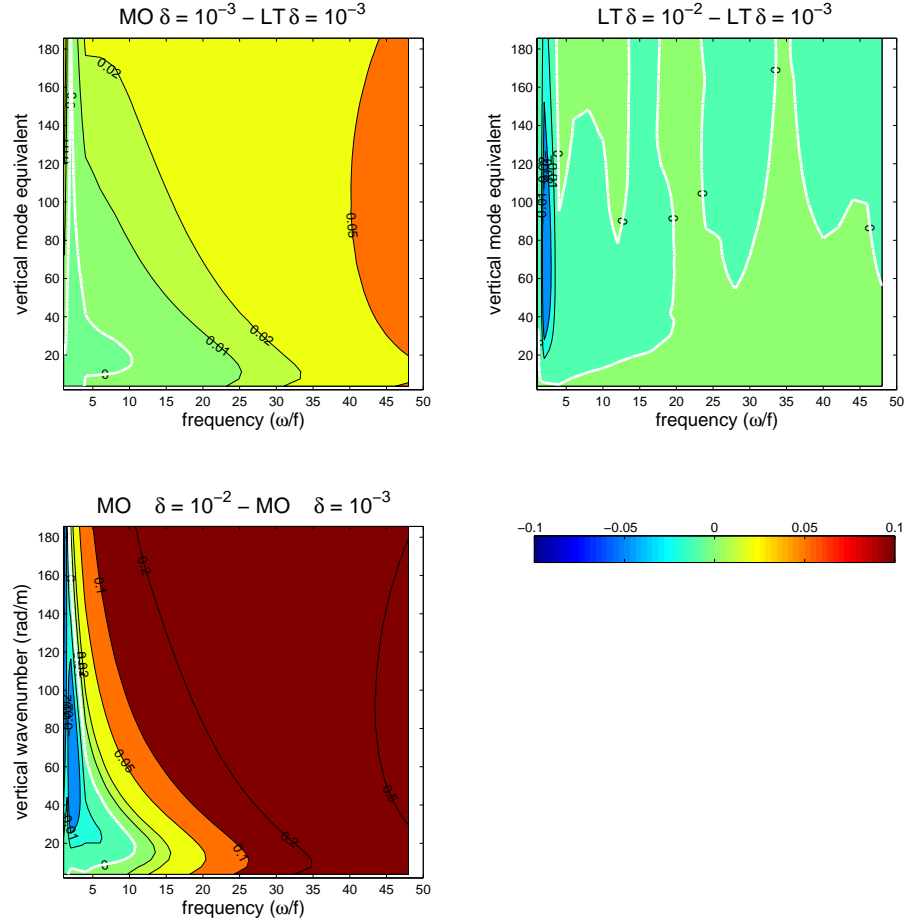


Figure 7: Differences between nonlinearity parameter (3) calculated via Müller and Olbers (1975) and Lvov and Tabak (2004) with $\delta = 10^{-3}$, i.e. between Run I and Run II, (a), calculated with Lvov and Tabak (2004) with $\delta = 10^{-2}$ and $\delta = 10^{-3}$, i.e. the difference Run I and Run III (b), and finally between Müller and Olbers (1975) with $\delta = 10^{-2}$ and $\delta = 10^{-3}$, i.e. between Run II and Run IV (c).

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